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Ted Chihara and his work on orthogonal polynomials

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1. Who is T.S. Chihara?

Theodore Seio Chihara is well known in the research community of orthogonal polynomial and special functions, not only because of his influential book *An Introduction to Orthogonal Polynomials* [C18] but also because of his important research contributions. Chihara's bibliography is presented at the end of this paper [C1–C37]. He prepared his Ph.D. thesis *Generalized Hermite polynomials* at Purdue University under supervision of Arthur Rosenthal and obtained his degree in 1955. An extract of this thesis is published in [C1]. He held positions at Seattle University (Seattle, Washington), the University of Alberta (Edmonton, Canada), the University of Victoria (British Columbia, Canada), and Purdue University, where he taught at Calumet Campus (Hammond, Indiana) until he retired in 1998 after 26 years of service. He and his wife Amy (who unfortunately passed away a few years ago) have five children (Laura, Lisa, Linda, Jerry, and Gregg), and five grandchildren (Stephanie, Alex, Amy, Robert, and Jeremy). His oldest daughter Laura is also a mathematician who obtained her Ph.D. on combinatorial aspects of orthogonal polynomials under supervision of Dennis Stanton. Ted and Laura wrote one joint paper [C29] which pleased Ted very much. Laura provided us with the following information:

> "Now that he is retired, he no longer has to partake in his least favorite activity in academia: grading papers (well, nobody likes doing that!). He is somewhat of a movie buff and knows a lot about old movies, character actors, etc. He has quite an extensive video collection at home — he is always taping old movies from the television. Now that he is retired, he goes frequently to the movies with his good friends the Johnsons and Merk. He likes to babysit his

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grandchildren. He was even contemplating taking up roller-blading, though I'm pretty sure he hasn't pursued that idea in awhile!"

Laura Chihara

Ted Chihara has two brothers (Charles and Paul) and one sister (Cathy). Charles Chihara is a professor at the University of California, Berkeley, and his interest is in philosophy of mathematics and logic. Paul Chihara is a composer who composed many concert works and music for over 70 motion pictures and series for television. Cathy Chihara is a music teacher at a junior high school in the Los Angeles area. His two brothers provided us with the following stories about Ted:

"*Ted's Notes*

In my first term as a graduate student in mathematics at Purdue, I took Professor Rosenthal's *Theory of Functions of a Real Variable*. At some point in the term, I was unclear about something in his lectures, so I asked Ted (who was writing his thesis under Rosenthal's direction) if he still had any notes from the time he took that course. He handed me a neatly bound copy of his notes of the course he had taken some three years earlier. What I had before me was amazing: these notes were beautifully hand printed in ink, complete with diagrams, problem sets worked out, and examinations taken. I had to look twice to make sure that what he had given me was not a publication. Indeed, I had the feeling that photocopies of the notes could be used, with very little change, as a text book. Later, I discovered that the notes he took in practically all his mathematics courses were recopied with the same attention to mathematical accuracy, logical rigor and devotion to esthetics that were exhibited in notes for the analysis course. It was clear that producing these notes was, for Ted, a labor of love and that, not unlike the monks of old, he genuinely treasured the mathematics that had been revealed to him, like Divine truths, by his professors."

Charles Chihara

"*My brother Ted, and Satie*

I was 15 years old, practicing the violin for three hours every morning, and determined to succeed as a concert violinist or composer. That summer (1954), in Seattle, I had the rare luxury and pleasure of being with my eldest brother Ted, who was working on his Ph.D. thesis. Several years earlier, he had gone off to Purdue University, to pursue his doctoral studies, and I seldom saw him. He was also nine years older than I, so in many respects we were strangers to each other. But that summer we bonded deeply and permanently, and we owe that special friendship to, among others, the nineteenth century French composer Eric Satie. Seattle in those days was culturally quite provincial. There were no major opera, ballet, or symphony organizations in the entire Pacific Northwest, and most serious music students went east (which usually meant New York City) to study. Indiana seemed to me a very close proximity. And Ted brought home with him the rare treasures of the then brand new invention, the Long Playing records. Among his musical offerings were the eccentric *Three Pieces in the Shape of a Pear* by Satie, symphonies by Beethoven in marvelous performances, piano concerti by Brahms, Saint-Saens, and Rachmaninoff, as well as works by Handel and Haydn. Every morning while I was practicing, Ted would be working on his mathematics, often in the same living room. After putting away my violin, I would listen to his records, while he went on writing. Together we traveled through miles of Mozart and pages of thesis. I learned to love those pieces, all of which were new to me (I had been brought up on popular music, and often performed it in a traveling USO troupe). Years later, I still love those pieces that Ted introduced me to in those halcyon years back in Seattle. I have since become a professional composer, and Ted a respected scholar. But for me, I will always think of us as kids exploring the worlds of music and mathematics together, and cherish that summer of 1954."

Paul Chihara

2. Chihara's work on orthogonal polynomials

Ted spent his whole professional career at schools, where teaching was the first and often the only priority, with heavier teaching loads than most people active in research have. It is a mark of his quality that he continued to do interesting research over his whole career. Ted's work in orthogonal polynomials has three main parts. One is on general orthogonal polynomials and their three term recurrence relations. A second is on specific sets of orthogonal polynomials, where weight functions can be found explicitly. The third is his book.

2.1. General orthogonal polynomials

In his second paper [C2], published in 1957, Chihara starts with a given sequence of orthogonal polynomials satisfying a recurrence relation

$$P_n(x) = (x + b_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 2, 3, \ldots \tag{2.1}$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x + b_1$, where b_n is real and $\lambda_n > 0$. The co-recursive polynomials $P_n^*(x)$ are then defined to satisfy the recurrence relation

$$P_n^*(x) = (x + b_n)P_{n-1}^*(x) - \lambda_n P_{n-2}^*(x), \quad n = 2, 3, \ldots \tag{2.2}$$

but with initial conditions $P_0^*(x) = 1$ and $P_1^*(x) = P_1(x) - c$. Hence the co-recursive polynomials $P_n^*(x)$ have the same recurrence coefficients as the polynomials $P_n(x)$, except for the first coefficient b_1 which is changed to $b_1 - c$. The problem then is to obtain properties of the $P_n^*(x)$ from those of $P_n(x)$. This amounts to a small perturbation of the recurrence coefficients (only one coefficient is changed). Later this was extended by also changing the first coefficient λ_2 to $c'\lambda_2$ (co-dilated polynomials) or a combination of both (co-modified polynomials, see [17] and references there; Allaway considered these already in his thesis [2] but the results were never published). In 1961, Wendroff [29] generalized Chihara's result. Wendroff started with two polynomials $G_{s-1}(x)$ and $G_s(x)$ of degrees $s - 1$ and s, respectively, with real and simple zeros so that the zeros of $G_{s-1}(x)$ and $G_s(x)$ interlace. He then showed how to construct a system of orthogonal polynomials $\{p_n(x)\}$ such that $p_{s-1}(x) = G_{s-1}(x)$ and $p_s(x) = G_s(x)$ and the p_n's satisfy any given recurrence relation of the type (2.1) for $n > s$. In 1957, the same year when [C2] appeared, Geronimus published a construction similar to Wendroff's but only for the Chebyshev case, that is he used (2.1) with $b_n = 0$ and $\lambda_n = \frac{1}{4}$ for $n > s$, see p. 52 in [12] for details and references. Geronimus also gave the distribution function. Geronimus' work went unnoticed in the west until the appearance of [12] in 1977. For an application of co-recursive polynomials in potential scattering, see [22]. A more general construction is to change a finite number of recurrence coefficients [21], and limiting situations where all the recurrence coefficients are changed slightly. Interesting situations are when

$$\lim_{n\to\infty} |b_n - b_n^*| = 0, \quad \lim_{n\to\infty} |\lambda_n - \lambda_n^*| = 0,$$

nowadays known as compact perturbations [20], and

$$\sum_{n=1}^{\infty} |b_n - b_n^*| < \infty, \quad \sum_{n=2}^{\infty} |\lambda_n - \lambda_n^*| < \infty,$$

nowadays known as trace class perturbations [26]. Compact perturbations where probably first studied by Blumenthal in his dissertation [6] (supervised by Hilbert), who basically studied the case

$$\lim_{n\to\infty} b_n = b, \quad \lim_{n\to\infty} \lambda_n = \lambda > 0. \tag{2.3}$$

Blumenthal's theorem states that the zeros of the polynomials $P_n(x)$ in (2.1) for which the recurrence coefficients satisfy (2.3) are dense on the interval $[-b-2\sqrt{\lambda}, -b+2\sqrt{\lambda}]$. Chihara knew about Blumenthal's work and pointed out this important result to many researchers who were unaware of it. Blumenthal also stated that the zeros have at most finitely many accumulation points outside the interval $[-b-2\sqrt{\lambda}, -b+2\sqrt{\lambda}]$, but Chihara showed that this is not true [C10] by giving a counterexample. Blumenthal's theorem may have been the first step towards more general results in operator theory, a part of mathematics which was developed in the first half of the 20th century. Compact perturbations preserve the essential spectrum (Weyl's theorem) and trace class perturbations of an absolutely continuous spectrum preserve absolute continuity (Kato–Rosenblum) [26,18,8,9]. Chihara made various contributions to the spectral theory of orthogonal polynomials and their corresponding Jacobi matrices, such as studying the essential spectrum (the derived set of the spectrum) [C12,C14], the number of mass points outside the essential spectrum [C21,C25] and the spectral theory in case of unbounded recurrence coefficients [C23,C33,C34].

A very important contribution of Chihara was his consistent use of chain sequences for the analysis of orthogonal polynomials. A sequence a_n $(n=1,2,3,\dots)$ is a (positive) chain sequence if there exists a second sequence g_n $(n=0,1,2,\dots)$ such that

$$0 \leqslant g_0 < 1, \quad 0 < g_n < 1, \quad n \geqslant 1 \tag{2.4}$$

$$a_n = (1-g_{n-1})g_n, \quad n \geqslant 1. \tag{2.5}$$

We have not been able to determine who first used the term chain sequence but the first systematic exposition seems to be in Wall's book [27] published in 1948. A few years earlier Wall and Wetzel [28] introduced the concept of maximal and minimal parameter sequences of a chain sequence. It must be noted that many earlier results of Stieltjes, Pringsheim and Van Vleck can be easily expressed in terms of chain sequences. Chihara used chain sequences in several of his papers and in his book and it was certainly Chihara's efforts that made chain sequences a standard tool in the analysis of orthogonal polynomials and their extreme zeros [C4,C10,C32,C34]. A first observation [C4,C10,C18] is that for a sequence of orthogonal polynomials with recurrence relation

$$P_n(x) = (x-c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x),$$

with $P_{-1}(x)=0$ and $P_0(x)=1$, the sequence

$$a_n = \frac{\lambda_{n+1}}{(c_n - x)(c_{n+1} - x)}$$

is a chain sequence if $x \leqslant \xi_1$ or $x \geqslant \eta_1$, with minimal parameter sequence

$$m_n = 1 - \frac{P_{n+1}(x)}{(x - c_{n+1})P_n(x)}.$$

Here we have used the quantities

$$\xi_i = \lim_{n\to\infty} x_{n,i}, \quad \eta_j = \lim_{n\to\infty} x_{n,n-j+1},$$

where $x_{n,1} < x_{n,2} < \cdots < x_{n,n-1} < x_{n,n}$ are the zeros of $P_n(x)$. These quantities clearly exist since the interlacing property of the zeros of orthogonal polynomials implies that $x_{n,i}$ is a decreasing sequence in n and $x_{n,n-j+1}$ is an increasing sequence in n. Furthermore the limiting sequence ξ_i is increasing (if $\xi_1 \neq -\infty$) and η_j is decreasing (if $\eta_1 \neq \infty$). These quantities are used very often by Chihara and their importance is in obtaining properties of the support of the orthogonality measure. Chain sequences are very useful to get information on the largest and smallest zeros of orthogonal polynomials and on the true interval of orthogonality. It is interesting to note that one easily discovers chain sequences by row reducing a finite tridiagonal matrix. Let $a_i \neq 0, 1 \leqslant i \leqslant n$ and consider a tridiagonal matrix A with entries $b_j\delta_{i,j} + a_j\delta_{i,j+1} + a_i\delta_{i,j-1}, 1 \leqslant i,j \leqslant n$. The pivots are $b_1, b_2 - a_1^2/b_1, b_3 - a_2^2/(b_2 - a_1^2/b_1), \ldots$. The pivots are positive if and only if the b's are positive and $a_1^2/b_1b_2, a_2^2/[b_2b_3(1 - a_1^2/b_1b_2)], \ldots$ are in $(0,1)$. This amounts to requiring that $\{a_j^2/b_jb_{j+1}\colon 1 \leqslant j < n\}$ is a chain sequence. The parameters are $g_0 = 0$, $g_1 = a_1^2/b_1b_2, \ldots$. Thus, A is positive definite if and only if $b_i > 0, 1 \leqslant i \leqslant n$ and $\{a_j^2/b_jb_{j+1}\colon 1 \leqslant j < n\}$ is a chain sequence. It is clear that the eigenvalues of A are in (a,b), if and only if both $A - aI$ and $bI - A$ are positive definite. This gives a characterization, in terms of chain sequences, of requiring the eigenvalues to lie in (a,b). For some recent papers using chain sequences we refer the interested reader to [13,24,25].

2.2. Specific sets of orthogonal polynomials

Many of Chihara's papers come from various characterization theorems. The first was his determination [C9] of orthogonal polynomials whose generating function has the form

$$\sum_{n=0}^{\infty} P_n(x)w^n = A(w)B(xw),$$

which is known as a Brenke type generating function. There is an interesting new set of orthogonal polynomials which Chihara found, Case II in the nonsymmetric case. Later in [C13] he found the orthogonality in this case. Not much has been done with these polynomials, which should be called Chihara polynomials. One point about these polynomials is that the measure has mass on the whole real line and yet is not symmetric. This was not the first example: Meixner probably found the first explicit example [19], but there are not many known cases. These Chihara polynomials are linear combinations of other orthogonal polynomials

$$V_{2m}(x) = W_m(q,b;x^2) + (1 - q^m)W_{m-1}(q,bq;x^2),$$

$$V_{2m+1}(x) = xW_m(q,bq;x^2) + (1 - b)W_m(q,b;x^2),$$

where $W_n(q,b;x)$ are Wall polynomials. They are also described in his book [C18, p. 167, system (H)] and were studied further in the joint paper with Laura Chihara [C29].

A famous example of an indeterminate moment problem is given by the lognormal weight $\exp(-\gamma^2 \log^2 x)$ on $[0, \infty)$. The corresponding orthogonal polynomials are known as Stieltjes–Wigert polynomials. Chihara gave other orthogonality measures with the same moments in [C11] and for generalized Stieltjes–Wigert polynomials in [C19]. All these measures are discrete. Roy Leipnik [16] rediscovered some of Chihara's results. Later, Chihara and Ismail [C36] studied other indeterminate moment problems and they were able to find extremal solutions of the moment problem for the indeterminate cases of the orthogonal polynomials found by Chihara and Al-Salam in [C17], which will be described in the next paragraph.

Chihara and Al-Salam wrote a nice paper [C16] containing a new characterization of the classical orthogonal polynomials similar to the famous ones found by Bochner (which was probably given first by Sonine), Hahn and Tricomi. The (very) classical orthogonal polynomials of Jacobi, Laguerre, and Hermite (and even the Bessel polynomials) are characterized as the only orthogonal polynomials with a differentiation formula of the form

$$\pi(x)P_n'(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x),$$

where $\pi(x)$ is a polynomial. There is likely to be a similar theorem with the derivative replaced by any of the finite-difference operators which act on classical type polynomials. These include a finite difference operator, a q-difference operator, and possibly some divided difference operators. As far as we know, no one has tried to see if the ideas in [C16] can be used in these more general settings.

The paper Chihara and Al-Salam wrote on convolutions of orthogonal polynomials [C17] is very nice. It was done before a more general class of orthogonal polynomials was found (the Askey–Wilson polynomials). The problem Al-Salam and Chihara considered is to find all families of orthogonal polynomials $p_n(x)$ and $q_n(x)$ such that their convolution

$$Q_n(x, y) = \sum_{k=0}^{n} p_k(x) q_{n-k}(y)$$

also gives a family of orthogonal polynomials in x. All orthogonal polynomials with a generating function of the form

$$\sum_{n=0}^{\infty} P_n(x) w^n = A(w) \mathrm{e}^{xB(w)},$$

(Meixner–Sheffer generating function) satisfy this convolution property. Chihara and Al-Salam found a new family, which was later named the Al-Salam–Chihara polynomials in [4]. Their weight function was found later by Askey and Ismail [4] using continued fractions and Markov's theorem and by Askey and Wilson [5] as a special case of the Askey–Wilson polynomials. We now understand that the Al-Salam–Chihara polynomials correspond to the Laguerre polynomials in the Askey scheme [15] and they share many of the interesting properties of the Laguerre polynomials. In this sense, the characterization theorem of Al-Salam and Chihara is a q-analogue of Meixner's characterization theorem. The continuous q-Hermite polynomials are a special case of the Al-Salam–Chihara polynomials and their weight function was found in [2] and [C17]. The moment problem has a unique solution if $0 < q < 1$ but when $q > 1$ there are cases when the moment problem has infinitely many solutions. The latter case has been investigated in [4] and the elements of the Nevannlina matrix were computed in [C36].

A related paper is the one on q-Pollaczek polynomials [C28], also written by Al-Salam and Chihara, where a few new families of orthogonal polynomials were found with a generating function of a special type, which are counterexamples of a conjecture of Andrews and Askey.

The paper by Chihara and Ismail [C24] contains one of the strangest weight functions ever seen. The motivation came from a paper by van Doorn [10] on a queueing process where potential customers are discouraged by queue length. The polynomials are a natural nonsymmetric extension of work of Carlitz [7] and of Karlin and McGregor [14], but the weight function is something else. The support is a (nonsymmetric) discrete set with one accumulation point at zero, and 0 has no jump. Surprisingly, the mass points and their masses were found explicitly. The orthogonality of the symmetric case, as shown by Carlitz in [7], is equivalent to a formula of Euler which is usually proved through Lagrange inversion. This suggests that there is a Lagrange inversion theorem buried in the orthogonality of these polynomials. Otto Ruehr, in a private communication to Ismail, has given a proof that the total mass of the measure is equal to unity but this is far from the full orthogonality.

2.3. Chihara's book on orthogonal polynomials

As Chihara knows, one of us (Askey) read this book for the publisher, and recommended publication. Initially it did not sell well, partly because of the high price, and partly because there was not that much interest in orthogonal polynomials. The latter changed, and Chihara's book has proved to be very useful. Part of the use is the first two thirds, where he did things which were not in Szegő's book [23], such as proving Favard's theorem, writing about the connection with continued fractions, and a good deal about what can be said about orthogonal polynomials from knowledge of the recurrence relation, such as Krein's theorem characterizing when the support of the orthogonality measure has a finite set of accumulation points in terms of compact operators (Chihara uses the terminology *completely continuous operators*). The other reason his book has been useful is that the last two chapters gave a very nice survey of what was known about orthogonal polynomials which could be found explicitly. Much of this is now dated, but only because of new work, some of which was started because a reference in one of these two chapters led someone to ask how to extend the results mentioned there.

A review of Chihara's book (by R. Askey)

At the Bar-Le-Duc meeting on orthogonal polynomials Gautschi and Hahn remarked on the book review of Chihara's book which appeared in Zentralblatt für Mathematik. They each thought it was a poor review. I had missed it, but after returning to Madison looked it up. The reviewer read a different book than I did. Here is my opinion.

A couple of reviewers of this book think the reason for the renewal of interest in orthogonal polynomials is because of work in approximation theory and numerical analysis. This is false. Splines have taken over the role that polynomials used to play in approximation theory, and while orthogonal polynomials and Gaussian quadrature play a small role in the use of splines, this would not account for the wide interest in orthogonal polynomials. Actually there are a number of reasons for the interest in orthogonal polynomials. General orthogonal polynomials are primarily interesting because of their three term recurrence relation. The first four chapters of Chihara's book treat orthogonal polynomials from the point of view of their recurrence relation, and what conditions on the coefficients imply about the weight function. After treating the standard general theorems, the author uses some results of Stieltjes to study the case when the measure is supported on a half line. These methods, which the author uses in the form of chain sequences, are powerful enough so the author is able to construct a set of orthogonal polynomials whose recurrence relation is

$$2xp_n(x) = p_{n+1}(x) + c_n p_{n-1}(x)$$

with $c_n = 1 + O(n^{-2})$, and the measure has an absolutely continuous part on $[-1, 1]$ and infinitely many mass points outside $[-1, 1]$. Others had claimed there could only be finitely many mass points outside $[-1, 1]$ when $c_n = 1 + O(n^{-2})$. This claim was in print three years before this book appeared, and the author knew his example a couple of years before publication of this book. The reason this is not included is that the book was accepted for publication by the publisher in 1970 (or 1971), but was not published until 1978. Some interesting work was done in this interval, but the deeper work of Nevai and his coworkers did not start to appear until 1979. Nevai's work is one of the real reasons there is a lot of work being done on general orthogonal polynomials. The problems were always interesting; what was missing were ideas about how to attack them. The first two thirds of Chihara's book is still a nice introduction to general orthogonal polynomials, but it is not nearly as good a summary of known methods as it was when it was written about 15 years ago. [1]

The last two chapters summarize some of what Chihara knew about explicit sets of orthogonal polynomials. Chihara knows the literature on specific sets of orthogonal polynomials very well, and these two chapters are a gold mine for those of us who want to study specific sets of orthogonal polynomials. There are a few sets that were known in 1970 that are not mentioned; The most important was found by Rogers in 1895, and the only mention of these polynomials is in a cryptic reference to Allaway's thesis, where these polynomials were found for the fourth time. However in the early 1970s no one understood these polynomials, so it is not surprising they were not included. Many of the polynomials given in these two chapters can now be put into the chart of classical hypergeometric orthogonal polynomials (those on the Tableau d'Askey) or in the basic hypergeometric version of this table which has not been made yet. [2] However there are many nonclassical polynomials given in these two chapters, and some of them, like the Pollaczek polynomials, are very important.

In summary, a much better version of this book could be written now, but it was written in the late 1960s and even when it was published in 1978 it was an excellent summary of what the author set out to do. And what he wanted to do was the right thing to do at that time. He did not treat much of the material in Freud's book on orthogonal polynomials [11], there was no need to. Together these two books are a good supplement to, but not a replacement for, Szegő's great book on orthogonal polynomials [23].

3. Ted Chihara, from the eyes of a graduate student

Ted Chihara visited the University of Alberta during the academic year 1969–1970. At the time both Mourad Ismail and William Allaway were graduate students working with Waleed Al-Salam.

[1] This review was written in 1984, so 15 years ago refers to the end of the 1960s.

[2] By now, this basic hypergeometric extension has been made. See [15].

Al-Salam and Chihara conducted a year long classical analysis seminar which was very well-attended. Chihara and other participants lectured on topics related to the moment problem from [1] and other sources. The lecturing styles of Al-Salam and Chihara were very different. Chihara lectured on moment problems, continued fractions and the general theory of orthogonal polynomials and, every once in a while, pulled a chain sequence out of thin air and solved a problem that had earlier puzzled the audience. Chihara's lectures and use of chain sequences made an everlasting impression on Ismail, who later used chain sequences on several occasions.

Chihara's paper [C12], which deals with conditions under which a point is a limit point of the spectrum, was written at that time. Chihara's paper settled a conjecture of D. Maki and led Allaway [3] to do further work on this problem.

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